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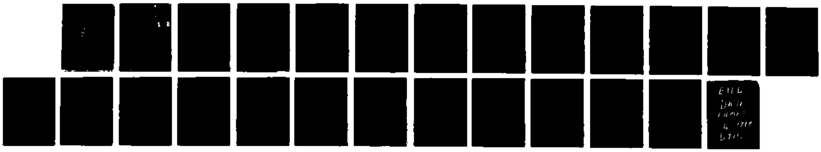

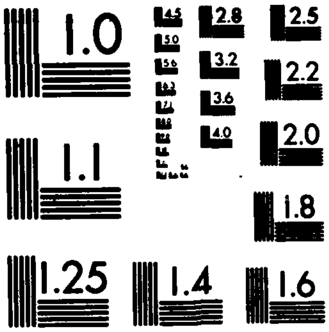

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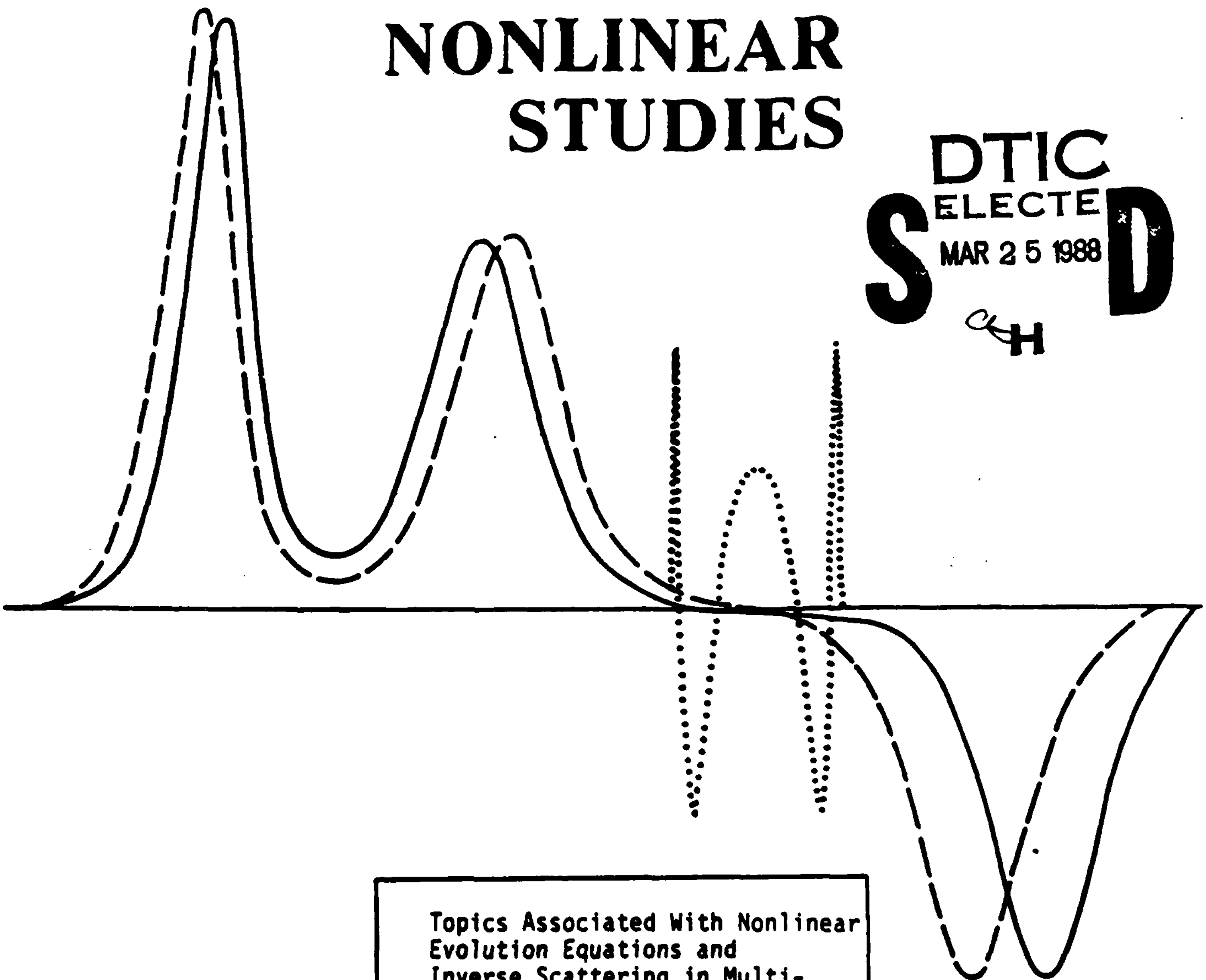

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Topics Associated With Nonlinear Evolution Equations and Inverse Scattering in Multi-dimensions

by

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N00014-86-K-0603

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March 1987

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TOPICS ASSOCIATED WITH NONLINEAR EVOLUTION EQUATIONS AND INVERSE SCATTERING IN MULTIDIMENSIONS

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Abstract

In recent years the basic structure required to implement the inverse scattering transform in 1+1 and 2+1 dimensions has been clarified and extended. Aspects involved with fully multidimensional problems have also been treated. In particular the inverse scattering associated with various multidimensional operators and generalizations of the Sine-Gordon and self-dual Yang-Mills equations have been studied. A review of some of this work will be discussed in this review.

The Inverse Scattering Transform (I.S.T.) is a method to solve certain nonlinear evolution equations. There has been wide ranging interest in this method for many reasons. A review of earlier work can be found in [1]. A surprisingly large number of physically interesting nonlinear equations can be solved via IST; there are many applications in physics including: surface waves, internal waves, lattice dynamics, plasma physics, nonlinear optics, particle physics and relativity. Mathematically speaking the field is also quite rich, with nontrivial results in the areas of analysis, group theory, algebra, differential and algebraic geometry being used by various researchers. From ~~our~~ the authors point of view, IST allows us to solve the Cauchy problem for these nonlinear systems. ~~We shall~~ This document concentrates on questions in infinite space. All of the nonlinear equations discussed below arise as the compatibility condition of certain linear equations, one of which is identified as a scattering (direct and inverse scattering is required) problem and the other(s) serves to fix the "time evolution" of the scattering data.

In one spatial dimension the prototype problem is the (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

The KdV equation is compatible with

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$$v_{xx} + u(x,t)v = \lambda v \tag{2}$$

$$v_t = (\gamma + u_x)v - (4\lambda + 2u)v_x \tag{3}$$

i.e. $v_{xxt} = v_{txx}$ implies (1). Equation (2) is the time independent Schrodinger scattering problem, λ the eigenvalue (γ = const. in (3)). The solution of (1) on the line: $-\infty < x < \infty$ for initial values $u(x,t=0)$ vanishing sufficiently rapidly at infinity is obtained by studying the

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associated direct and inverse scattering problem of (2) and using (3) to fix the time evolution of the scattering data. It turns out that the inverse problem amounts to solving a matrix Riemann-Hilbert boundary value problem (RHBVP) whose jump discontinuity depends explicitly on the scattering data. Calling $\lambda=-k^2, v(x,k)=u(x,k)e^{-ikx}$ the RHBVP takes the following form,

$$(\mu_+-\mu_-)(x,t,k) = \mu_-(x,t,\alpha(k))\, V(x,t,k) \text{ on } \Sigma$$
$$\mu_\pm\to 1,\ |k|\to\infty \qquad (4)$$

where

$V(x,t,k) = r(k,t)\, e^{2ikx}$, $\alpha(k) = -k$, $\Sigma=\{k: k\in\mathbb{R}\}$, and $\mu_\pm$ are the limiting boundary values, as $\mathrm{Im}k\to 0\pm$, of meromorphic functions in the upper (+) lower (-) half plane. (4) may be converted into a linear integral equation by taking a minus projection and the potential is reconstructed via

$$u(x,t) = -\frac{1}{\pi}\frac{\partial}{\partial x}\int_C \mu(k,x,t,-k)\, V(x,t,k)dk \qquad (5)$$

where the contour is taken above all poles of $r(k,t)$; of which there is at most a finite number, $k_j = i\kappa_j,\ \kappa_j>0 \quad j = 1,\cdots N$. The scattering data: the reflection coefficient, $r(k,t)$ evolves simply in time

$$r(k,t) = r(k,0)\, e^{8ik^2t} \qquad (6)$$

The above scheme may be extended so as to solve a surprisingly large number of interesting nonlinear evolution equations. There are two scattering problems of particular interest in one dimension:

(i) Scalar scattering problems:

$$\frac{d^nv}{dx^n} + \sum_{j=2}^{n} u_j(x)\,\frac{d^{n-j}v}{dx^{n-j}} = \lambda v,$$

$$v(x,k),\quad u_j \in \mathbb{C}$$

(ii) First order systems - generalized AKNS

$$\frac{dv}{dx} = i\,k\,J\,v + q\,v$$

$$v(x,k), q(x) \in \mathbb{C}^{N\times N},\ J = \mathrm{diag}\,(J^1,\cdots J^n)$$
$$J^i\neq J^j, i\neq j$$

$$q^{ii} = 0.$$

Via an appropriate transformation the inverse problem associated with (i), (ii) can be expressed as a matrix RHBVP of the form (4). The potentials u_j, q can be shown to satisfy nonlinear evolution equations by appending to (i) and (ii), suitable linear time evolution equations. One then finds that the scattering data $V(x,t,k)$ evolves simply in time. Well known solvable nonlinear equations include the Boussinesq, modified KdV, sine-Gordon, nonlinear Schrodinger, and three wave interaction equations. The reader may wish to consult for example [2a-e] for a detailed discussion of some of this material.

It is most significant that these concepts can be generalized to 2 spatial plus one time dimension. Here the prototype equation is the Kadomtsev-Petviashvili (K-P) equation:

$$(u_t + 6uu_x + u_{xxx})_x = -3\sigma^2 u_{yy} \tag{7}$$

which is the compatibility equation between the following linear problems:

$$\sigma v_y + v_{xx} + u(x,y,t)v = 0 \tag{8}$$

$$v_t + 4v_{xxx} + 6uv_x + 3\left(u_x - \sigma \int_{-\infty}^{x} u_y dx'\right)v + \gamma v = 0 \tag{9}$$

(γ = const.). We shall consider the question of solving (7) for $u(x,y,0)$ decaying sufficiently rapidly in the plane $r^2 = x^2 + y^2 \to \infty$. Physically speaking, both cases $\sigma^2 = -1$ (KPI) $\sigma^2 = +1$ (KPII) are of interest. Whereas KPI can be related to a RHBVP of a certain type (nonlocal; see ref.[3]) KPII turns out to require new ideas. Letting

$$v = \mu(x,y,k)e^{ikx + k^2 y/\sigma}$$

$\sigma = \sigma_R + i\sigma_I$, $\sigma_R \neq 0$. Then there exist functions μ bounded for all x,y satisfying $\mu \to 1$ as $|k| \to \infty$. However such a function turns out to be nowhere analytic in k, rather it depends nontrivially on both the real and imaginery parts of $k (k = k_R + ik_I)$. $\mu = \mu(x,y,k_R,k_I)$.

In fact μ satisfies a generalization of a RHBVP - namely a $\bar{\partial}$ (DBAR) problem where μ satisfies,

$$\frac{\partial\mu}{\partial \bar{k}} = \mu(x,y,\xi_0,k_I)\ V(x,y,k_R,k_I) \tag{10}$$

where $\frac{\partial}{\partial \bar{k}} = \frac{1}{2}(\frac{\partial}{\partial k_R} + i\ \frac{\partial}{\partial k_I})$ and V has the structure

$$V(x,y,k_R,k_I) = \frac{sgn(k_0)e^{i\beta(x,y,k_R,k_I,\xi_0)}}{2\pi|\sigma_R|} T(k_R,k_I)$$

$$\beta(x,y,k_R,k_I,\xi_0) = (x + 2y\frac{k_I}{\sigma_R})(\xi_0 - k_R) = -2(x + 2y\frac{k_I}{\sigma_R})k_0$$

$$\xi_0 = -k_R - \frac{2\sigma_I}{\sigma_R}k_I,\ k_0 = k_R + \frac{\sigma_I}{\sigma_R}k_I \tag{11}$$

(10-11) may be converted into a linear integral equation by employing the generalized Cauchy formula. $T(k_R,k_I)$ is viewed as the "nonphysical" data, (i.e. inverse scattering data or inverse data) and the potential is reconstructed via

$$u(x,y) = \frac{2i}{\pi}\frac{\partial}{\partial x}\iint \mu(x,y,\xi_0,k_I)V(x,y,k_R,k_I)dk_R dk_I. \tag{12}$$

The basic ideas used in order to derive these equations is as follows. We convert the equation for $\mu = \mu(x,y,k)$:

$$\sigma\mu_y + \mu_{xx} + 2ik\mu_x - u(x,y)\mu = 0 \tag{13}$$

into an integral equation

$$\mu(x,y,k) = 1 + \tilde{G}(u,\mu) \tag{14}$$

where

$$\tilde{G}(f) = G*f = \iint G(x-x',y-y',k)\ f(x',y')dx'dy', \tag{15}$$

the Green's function kernel being givenby $(k=k_R+ik_I)$:

$$\begin{aligned} G(x,y,k_R,k_I) &= \frac{1}{(2\pi)^2}\ \frac{e^{i(\xi x+\gamma y)}}{(i\sigma\eta-\xi^2-2k\xi)}d\xi d\gamma \\ &= \frac{sgn(y)}{2\pi\sigma}\int d\xi e^{ix\xi\ +\ \xi(\xi+2k)y/\sigma} \\ &\cdot\ \Theta\ \ (-y\sigma_R(\xi^2+2\xi k_0))d\xi \end{aligned} \tag{16}$$

where $k_0 = k_R - \frac{\sigma_I}{\sigma_R} k_I$ and $\sigma(x) = \{1 x>0,\ 0 x<0\}$ (16)

The $\bar{\partial}$ derivative of the Green's function is especially simple,

$$\frac{\partial G}{\partial \bar{k}}(x,y,k_R,k_I) = \frac{\text{sgn}(k_0)}{2\pi|\sigma_R|} e^{i\beta(x,y,k_R,k_I)} \tag{17}$$

when

$$\partial/\partial\bar{k} = \frac{1}{2}\left(\frac{\partial}{\partial k_R} + i\frac{\partial}{\partial k_I}\right) \text{ and}$$

$$\beta(x,y,k_R,k_I) = -2\left(x+2y\frac{k_I}{\sigma_R}\right)k_0.$$

Taking the $\bar{\partial}$ derivative of (14)

$$\frac{\partial \mu}{\partial \bar{k}}(x,y,k_R,k_I) = \iint \frac{\partial G}{\partial \bar{k}}(x-x',y-y',k_R,k_I)u(x',y')\mu(x',y',k_R,k_I)dx'dy'$$
$$+ \iint G(x-x',y-y',k_R,k_I)u(x',y')\frac{\partial \mu}{\partial \bar{k}}(x',y',k_R,k_I)dx'dy' \tag{18}$$

and using (17) shows that

$$\frac{\partial \mu}{\partial \bar{k}} = \frac{\text{sgn}(k_0)}{\pi\ \sigma} T(k_R,k_I)\ w(x,y,k_R,k_I) \tag{19}$$

where $T(k_R,k_I) = \iint e^{-i\beta(x,y,k_R,k_I)}u(x,y)\mu(x,y,k_R,k_I)dxdy$ and

$w(x,y,k_R,k_I)$ satisfies:

$$w(x,y,k_R,k_I) = e^{i\beta(x,y,k_R,k_I)} + \iint G(x-x',y-y',k_R,k_I)$$
$$u(x',y')w(x',y',k_R,k_I)dx'dy'. \tag{20}$$

Multiplying (20) by $e^{-i\beta(x,y,k_R,k_I)}$ and employing the following symmetry condition on the Green's function

$$e^{-i\beta(x,y,k_R,k_I)}G(x,y,k_R,k_I)$$
$$= G(x,y,\xi_0,k_I) \tag{21}$$

where $\xi_0 = -k_0 - \frac{\sigma_I}{\sigma_R}k_I$, yields

$$w(x,y,k_R,k_I) = e^{i\beta(x,y,k_R,k_I)}\ \mu(x,y,\xi_0,k_I) \tag{22}$$

whereupon (10-11) follow. The eigenfunction μ is recovered with the generalized Cauchy formula

$$\mu(x,y,k_R,k_I) = 1 + \frac{1}{\pi}\iint \frac{\frac{\partial\mu}{\partial\bar{k}}(x,y,k_R',k_I')}{k-k'}\,dk_R',dk_I' \tag{23}$$

noting that using (10-11), (23) becomes a linear integral equation for μ. The potential $u(x,y)$ is recovered by taking $k\to\infty$ in (13) or (14) and (23).

For the K-P the evolution of the data obeys ($\gamma = 4ik^3$ in (9)

$$\frac{\partial T}{\partial t} = (8ik_0)(6kk_0 - 4k_0^2 - 3k^2)T \tag{24}$$

where $k_0 = k_R + \frac{\sigma_I k_I}{\sigma_R}$, $k = k_R + ik_I$.

Special cases include $\sigma = \sigma_R + i\sigma_I$:

(a) KP_{II}; $\sigma = -1$: $\sigma_R = -1, \sigma_I = 0$

$$\frac{\partial T}{\partial t} = 8ik_R(3k_I^2 - k_R^2)T \tag{25}$$

(b) KP_I; $\sigma = i$: $\sigma_R \to 0-$, $\sigma_I = 1$, $\hat{k}_I = k_I/\sigma_R$

$$\frac{\partial T}{\partial t} = -8i(k_R + \hat{k}_I)(k_R^2 + 2k_R\hat{k}_I + 4\hat{k}_I^2)T \tag{26}$$

These formulae allow us in principle to solve the Cauchy problem for K-P and in particular the limit (ii) discussed above allows us to give an alternative solution for KP_I via $\bar{\partial}$ and not via a nonlocal RHBVP.

Similar ideas apply to higher order scalar problems

(iii) $$\sigma\frac{\partial v}{\partial y} + \frac{\partial^n v}{\partial x^n} + \sum_{j=2}^{n} u_j(x)\frac{\partial^{n-j} v}{\partial x^{n-j}} = 0$$

where: $v, u_j \in \mathbb{C}$ and to first order systems

(iv) $$\sigma\frac{\partial v}{\partial y} + J\frac{\partial v}{\partial x} + q(x,y)v = 0$$

where: $v, q \in \mathbb{C}^{N\times N}$, $J = \mathrm{diag}(J^1,\ldots,J^N)$, $J^i \neq J^j$, $i \neq j$ with $q^{ii} = 0$.

Interested readers may consult refernce 4a, and review 4b for more details.

The notion of $\bar{\partial}$ extends to higher dimensional scattering and inverse scattering problems. However as we shall mention, despite the fact that the inverse scattering problem is essentially tractable there does not appear to be any local nonlnear evolution equations in dimensions greater than 2 + 1 associated with multidimensional generalizations of (iii) or (iv).

Our prototype scattering problem will be

$$\sigma v_y + \Delta v + u(x,y)v = 0$$
$$\Delta = \sum_{\ell=1}^{n} \frac{\partial^2}{\partial x_\ell^2}, \qquad x \in \mathbb{R}^n, \quad y \in \mathbb{R} . \tag{27}$$

Letting

$$v = \mu(x,y,k)e^{ik\cdot x + k^2 y/\sigma}$$
$$k = k_R + ik_I, \quad k \in \mathbb{C}^n$$
$$k\cdot x = \sum_{1}^{n} k_j x_j, \quad \sigma = \sigma_R + i\sigma_I .$$

Then there exist functions μ bounded for all x, y satisfying $\mu \to 1$, as $|k_j| \to \infty$, j = 1,...,n. When $\sigma_R \neq 0$ μ turns out to be nonanalytic in each of the variables k, i.e. $\mu = \mu(x,y,k_{R_1},\dots k_{R_n}, k_{I_1},\dots,k_{I_n})$ and satisfies a $\bar{\partial}$ problem linear in μ, in each of the variables k_j; i.e. we shall show that μ satisfies an equation of the form,

$$\frac{\partial \mu}{\partial \bar{k}_j} = \bar{T}_j(\mu); \qquad j = 1,\dots,n \tag{28}$$

where $\bar{T}_j$ is an appropriate linear integral operator.

The basic idea in order to derive (28) follows a similar format to the two dimensional case described earlier. From the definition of $\mu(x,y,k)$ below (27) we see that it satisfies

$$\sigma\mu_y + \Delta\mu + 2ik\cdot\nabla\mu - u(x,y) = 0. \tag{29}$$

We convert to an integral equation

$$\mu = 1 + \tilde{G}(u\,\mu) \tag{30}$$

where the Green's function kernel is given by

$$G(x,y,k_R,k_I) = \frac{1}{(2\pi)^{n+1}} \iint \frac{e^{i(x\cdot\xi+y\eta)}}{i\sigma y-\xi^2-2k\cdot\xi} d\xi dy$$

$$= \frac{\mathrm{sgn}(y)}{\sigma} \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi+\frac{y}{\sigma}(\xi^2+2k\cdot\xi)}. \tag{31}$$

$$\cdot\ \Theta\left(-y\sigma_R(\xi^2+2\left(k_R+\frac{\sigma_I k_I}{\sigma_R}\right),\xi\right)d\xi. \tag{32}$$

Taking the $\bar{\partial}$ derivative of (30)

$$\frac{\partial\mu}{\partial \bar{k}_j} = \frac{\partial\tilde{G}}{\partial\bar{k}}(u\mu) + \tilde{G}\left(u\frac{\partial\mu}{\partial\bar{k}_j}\right), \tag{33}$$

and using

$$\frac{\partial G}{\partial \bar{k}_j}(x,y,k_R,k_I) = -\frac{1}{(2\pi)^n}|\sigma_R|\int e^{i\beta(x,y,k_R,k_I,\xi)}$$

$$\cdot(\xi_j-k_{Rj})\delta(\rho(\xi))d\xi \tag{34}$$

where

$$\beta(x,y,k_R,k_I\xi) = \left(x+2y\frac{k_I}{\sigma_R}\right).(\xi-k_R)$$

$$\rho(\xi) = \left(\xi+\frac{\sigma_I}{\sigma_R}k_I\right)^2 - \left(k_R+\frac{\sigma_I}{\sigma_R}k_I\right)^2 \tag{35}$$

shows that

$$\frac{\partial\mu}{\partial\bar{k}_j} = -\frac{1}{(2\pi)^n}\frac{1}{|\sigma_R|}\int T(k_R,k_I\xi)(\xi_j-k_{Rj})\delta(\rho(\xi))$$

$$\cdot\ w(x,y,k_R,k_I,\xi)d\xi \tag{36}$$

where

$$T(k_R,k_I,\xi) = \int e^{-i\beta(x,y,k_R,k_I,\xi)} u(x,y)\, w(x,y,k_R,k_I) dxdy \tag{37}$$

and w satisfies

$$w(x,y,k_R,k_I,\xi) = e^{i\beta(x,y,k_R,k_I\xi)} + \tilde{G}(uw). \tag{38}$$

Multiplying (37) by $e^{-i\beta}$ and using the symmetry condition

$$e^{-i\beta(x,y,k_R,k_I,\xi)}\, G(x,y,k_R,k_I) = G(x,y,\xi,k_I) \tag{39}$$

yields

$$w(x,y,k_R,k_I,\xi) = e^{-i\beta(x,y,k_R,k_I,\xi)} u(x,y,\xi,k_I) \tag{40}$$

and hence (36) gives

$$\frac{\partial \mu}{\partial \bar{k}_j} = \tilde{T}_j(\mu) = -\frac{1}{(2\pi)^n}\frac{1}{|\sigma_R|}\int T(k_R,k_I,\xi)(\xi_j-k_{Rj})$$
$$\cdot\ \delta(\rho(\xi))\, e^{i\beta(x,y,k_R,k_I,\xi)} \mu(x,y,\xi,k_I) d\xi. \tag{41}$$

We see that $\tilde{T}_j$ is an integral operator which depends on a scalar scattering function $T = T(k_R,k_I,\xi)\xi$ being effectively (n-1) integration parameters (due to the delta function in (41) in the nonlocal operator $\tilde{T}_j$).

One can use a generalized Cauchy formula such as (23) in order to obtain a linear integral equation to reconstruct μ. However due to the redundancy of the data discussed below, we find that an alternative method is more useful. The inverse problem is redundant, i.e. we are given $T(k_R,k_I,\xi)$ (3n-1 parameters) and we must reconstruct a local potential $u(x,y)$ (n+1 parameters). A serious issue is how to characterize admissible inverse data T, i.e. data that really arises from a local potential (small generic changes in $T(k_R,k_I,\xi)$ cannot be expected to arise from a local potential $u(x,y)$). Insight into this question is obtained by noting that T must satisfy a nonlinear constraint, one which is obtained by requiring $\partial^2\mu/\partial\bar{k}_i\partial\bar{k}_j = \partial^2\mu/\partial\bar{k}_j\partial\bar{k}_i$ $(i \neq j)$. the form of this constraint is given by

$$\mathcal{L}_{ij}(T) = \tilde{N}_{ij}[T] \qquad (42)$$

where $\mathcal{L}_{ij}$ is a linear operator and $\tilde{N}_{ij}$ a nonlinear (quadratic) nonlocal operator. These operators are given by

$$\mathcal{L}_{ij} = (\xi_j - k_{jR})\left(\frac{\partial}{\partial k_i} + \frac{1}{2}\frac{\partial}{\partial \xi_i}\right) - (\xi_i - k_{iR})\left(\frac{\partial}{\partial k_j} + \frac{1}{2}\frac{\partial}{\partial \xi_j}\right) \qquad (43)$$

$$\tilde{N}_{ij}(T) = \int [(\xi_j' - k_{jR})(\xi_i - \xi_i') - (\xi_i' - k_{iR})(\xi_j - \xi_j')]$$

$$\cdot \delta(\rho(\xi')\, T\, (k_R, k_I, \xi)\; T(\xi', k_I, \xi) d\xi'. \qquad (44)$$

There is, in fact, an explicit transformation of variables

$$(k_R, k_I, \xi) \to (\chi, w_0, w) \in \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}^n$$

which simplifies (42). Namely,

$$k_{R1} = \sum_{j=2}^{n} w_j \chi_{Rj} - \frac{w_1}{2} - \frac{\sigma_I w_0 w_1}{2w^2},$$

$$k_{Rj} = -w_1 \chi_{Rj} - \frac{w_j}{2} - \frac{\sigma_I w_0 w_j}{2w^2}, \quad (j \geq 2)$$

$$k_{i1} = \sum_{j=2}^{n} w_j \chi_{Ij} + \frac{\sigma_R w_0 w_1}{2w^2},$$

$$k_{ij} = -w_1 \chi_{Ij} + \frac{\sigma_R w_0 w_j}{2w^2}, \quad (j \geq 2)$$

$$\xi_1 = \sum_{j=2}^{n} w_j \chi_{Rj} + \frac{w_1}{2} - \frac{\sigma_I w_0 w_1}{2w^2},$$

$$\xi_j = -w_1 \chi_{Rj} + \frac{w_j}{2} - \frac{I w_0 w_j}{2w^2}. \quad (j \geq 2) \qquad (45)$$

transforms (42) into:

$$\frac{\partial T}{\partial \bar{x}_j} = \tilde{N}_{ij}(T)(x,w_0,w) \ . \ j=2,---n \tag{46}$$

using the generalized Cauchy formula (23) we have

$$I_j[T](x,w,w_0) = T(x,w,w_0) - \frac{1}{\pi}\iint \frac{\tilde{N}_{ij}(T)(\tilde{x},w,w_0)}{x - \tilde{x}} dx_R' dx_I'$$

$$= \hat{u}(w,w_0) \tag{47}$$

where

$$\tilde{x} = (x_2,x_3,---x_j',---x_n)$$

$$\hat{u}(w_0,w) = \iint e^{-i(yw_0+x\cdot w)}u(x,y)dxdy \tag{48}$$

We have used the fact that when $w_0 = 2k_I\cdot(\xi-k_R)/\sigma_R$ and $w = \xi-k_R$ are kept fixed, $T(x,w,w_0) \to \hat{u}(w,w_0)$ (The Fourier Transform of $u(x,y)$) for large $x_j(w_1 \neq 0)$; this is the analogue of the Born approximation.

We expect that for suitably "small" u (i.e. no homogeneous solutions to the relevant integral quations) if I is independent of x,j and decays sufficiently fast for $|w|,|w_0|\to\infty$, then $T(k_R,k_I,\xi)$ is admissable. Moreover (47) gives a formula to reconstruct the potential by quadratures. Limits to case $\sigma = i$ and reductions to stationary potentials $u(x,y) = u(x)$ can be carried out. Details can be found in Ref. [5a,b]. It should also be noted that in recent work Nachman and Lavine [5c] have extended theaw ideas to situations where there are homogeneous solutions to the relevant integral equations. (42) also suggests why simple local nonlinear evolution equations have not been associated with equation (27). Namely in the previous lower dimensional (2+1 and 1+1) problems the time evolution of the scattering data obeyed a particularly simple equation, (e.g. $\frac{\partial T}{\partial t} = \omega(k_R, k_I)T$). However in this case such a simple flow will not be maintained - due to the nonlinear constraint (42).

These ideas can be generalized to first order systems:

$$\text{(v)} \qquad \frac{\partial v}{\partial y} + \sigma \sum_{j=1}^{n} J_j \frac{\partial v}{\partial x_j} = qv$$

$v, q \in \mathbb{C}^{N\times N}, \quad J_j = \mathrm{diag}(J_j^1, \ldots, J_j^N)$

$J_j^k \neq J_j^\ell, \quad k \neq \ell .$

with many similar results obtained 6a,b,c; though there are some important differences as well: see ref. [6c]. Again the scattering data satisfies a nonlinear constraint. In general, there is no compatible local nonlinear evolution equation associated with (v). However when certain restrictions are put on J_j then the constraint equation becomes linear and the so-called N wave interaction equations are compatible with the system (v). Nachman and Ablowitz [6a] showed that at most, the system would be 3+1 dimensional, and Fokas[6b] showed that indeed the system is reducible to 2+1 dimensions by a transformation of independent variables (characteristic variables). In [6c] Fokas studies the inverse scattering of (v). For $\sigma = i$ he finds an equation similar to (42). However its integrated form shows that in order for the potential to be reconstructed one must solve a reduced system of equations of the form (v): i.e. for N = 2. This is in contrast to the scalar problem where reconstruction is via quadratures.

Beals and Coifman haven an alternative but similar formulation [7a,b] for multidimensional scalar problems.

There is an n-dimensional problem which also fits within the framework of IST: The so-called generalized wave and generalized sine-Gordon equation (GWE and GSGE). These equations arise in the context of differential geometry and serve to extend the classical results of Bäcklund for the sine-Gordon equation to n-dimensions [8]. The n-dimensional Bäcklund transformation is given by:

$$dX + XA^tX = A - XB, \tag{49}$$

where

$$dX = \sum_{j=1}^{n} \frac{\partial X}{\partial x_j} dx_j,$$

$$A_{ij} = \beta_i(z) a_{ij} dx_j,$$

$$B_{ij} = \frac{1}{a_{1i}} \frac{\partial a_{1j}}{\partial x_i} dx_j - \frac{1}{a_{1j}} \frac{\partial a_{1i}}{\partial x_j} dx_i, \quad 1 \le i, j \le n, \tag{50}$$

and $a = (a_{ij}) \in R^{n\times n}$. Equations (49-50) reduce to the Bäcklund transformation for the generalized sine-Gordon equation (GSGE) when

$$\beta_i(z) = (z^2 + (2\delta_{i1} - 1))/2z, \tag{51}$$

and for the generalized wave equation (GWE) when

$$\beta_i(z) = -(1-z^2)/2z \equiv \lambda(z). \tag{52}$$

The compatibility condition required for the existence of solutions to these Bäcklund transformations results in a system of second-order partial differential equations for an orthogonal $n \times n$ matrix $a = (a_{ij})$ in (49) which is a function of n independent variables $a = a(x_1, x_2, \ldots, x_n)$. The equation has the form

$$\frac{\partial}{\partial x_i}\left(\frac{1}{a_{1i}} \frac{\partial a_{1j}}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{a_{1k}} \frac{\partial a_{1i}}{\partial x_j}\right) + \sum_{k \ne i,j} \frac{1}{a_{1k}^2} \frac{\partial a_{1i}}{\partial x_k} \frac{\partial a_{1j}}{\partial x_k} = \epsilon a_{1i} a_{1j}, \quad i \ne j,$$

$$\frac{\partial}{\partial x_k}\left(\frac{1}{a_{1j}} \frac{\partial a_{1i}}{\partial x_j}\right) = \frac{1}{a_{1k} a_{1j}} \frac{\partial a_{1i}}{\partial x_k} \frac{\partial a_{1k}}{\partial x_j}, \quad i, j, k \text{ distinct},$$

$$\frac{\partial a_{jk}}{\partial x_k} = \frac{\partial a_{ji}}{a_{1i}} \frac{\partial a_{1k}}{\partial x_i}, \quad i \ne k, \tag{53}$$

where $\epsilon = 1$ for the GSGE and $\epsilon = 0$ for the GWE.

We observe that when $n = 2$ and $\kappa = 1$ (GSGE), the orthogonal matrix $a = \{a_{ij}\}$ given by

$$a = \begin{pmatrix} \cos\frac{1}{2}u & \sin\frac{1}{2}u \\ -\sin\frac{1}{2}u & \cos\frac{1}{2}u \end{pmatrix} \tag{54}$$

for the function $u = u(x,t)$ reduces the GSGE to the classical sine-Gordon equation ($\kappa = -1$),

$$u_{tt} - u_{xx} - \kappa \sin u = 0. \tag{55}$$

On the other hand when $n = 2$ and $\kappa = 0$, then with (54) the GWE reduces to the wave equation (55). When $n \geq 3$ the generalization of the wave equations discussed here is nonlinear.

The Bäcklund transformations (49) described above are in fact matrix Riccati equations. Linearizations of such a system can be performed in a striaghtforward manner. Introducing the transformation

$$X = UV^{-1}, \tag{56}$$

where U, V and n x n matrix functions of $x_1,\ldots,x_n$, the following linear

system is deduced:

$$\begin{pmatrix} dU \\ dV \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^t & B \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \tag{57}$$

with the components of A, B given by (50). Compatibility ensures that the orthogonal matrix $a = \{a_{ij}\}$ satisfies the GSGE with (51) and GWE with (52). Alternatively, if we call

$$\begin{pmatrix} U \\ V \end{pmatrix} = \psi \ , \tag{58}$$

the following linear system of 2n o.d.e.'s are obtained:

$$\frac{\partial \psi}{\partial x_j} = \lambda \tilde{A}_j \psi + C_j \psi, \tag{59}$$

where $\tilde{A}_j$, C_j are $2n \times 2n$ matrices with the block structure

$$A_j = \begin{pmatrix} 0 & \tilde{a}_j \\ \tilde{a}_j^t & 0 \end{pmatrix}, \quad C_j = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_j \end{pmatrix}. \tag{60}$$

Here $\tilde{a}_j$, $\tilde{\gamma}_j$ are $n \times n$ matrices having the following structure:

$$\begin{aligned} \tilde{a}_j &= (\tfrac{\delta}{\lambda} - 1)e_1 a_j + a_j, \\ a_j &= a e_j \end{aligned} \tag{61}$$

where $e_j = \{e_j\}_{ik}$ is the unit matrix

$$\{e_j\}_{ik} = \begin{cases} 1 & i = k = j, \\ 0 & \text{otherwise}, \end{cases} \tag{62}$$

and in component form γ_j takes the form

$$(\gamma_j)_{k\ell} = (1-\delta_{kj})\frac{1}{a_{\ell k}}\frac{\partial a_{\ell j}}{\partial x_k}\delta_{\ell j} - (1-\delta_{\ell j})\frac{1}{a_{1\ell}}\frac{\partial a_{1j}}{\partial x_\ell}\delta_{kj}. \tag{63}$$

In (61) a is the orthogonal matrix $R^n \to SO(n)$ associated with the GWE when $\delta = \lambda$ and with the GSGE when $\delta = \frac{1}{2}(z + 1/z)$, $\lambda = \frac{1}{2}(z - 1/z)$, and γ_j is the matrix (63): $R_n \to M_n(R)$, $\gamma_j + \gamma_j^t = 0$. Equations (53) arise as the compatibility condition associated with (58). More explicitly, for the GWE the scattering problem takes the form [$\psi = \psi(x,\lambda)$]

$$\frac{\partial \psi}{\partial x_j} = \lambda A_j \psi + C_j \psi \tag{64}$$

with

$$A_j = \begin{pmatrix} 0 & a_j \\ a_j^t & 0 \end{pmatrix}, \tag{65}$$

and C_j given by (60,63).

For the GSGE the scattering problem for $\psi = \psi(x,z)$ takes the form

$$\frac{\partial \psi}{\partial x_j} = \delta(z)\begin{pmatrix} 0 & e_1 a_j \\ a_j^t e_1 & 0 \end{pmatrix}\psi + \lambda(z)\begin{pmatrix} 0 & (1-e_1)a_j \\ a_j^t(1-e_1) & 0 \end{pmatrix}\psi + C_j\psi, \tag{66}$$

$\xi(z)$, $\lambda(z)$, C_j given above, or equivalently

$$\frac{\partial\psi}{\partial x_j} = \frac{z}{2} A_j\psi + \frac{z}{2} B_j\psi + C_j\psi, \tag{67}$$

where

$$B_j = \begin{pmatrix} 0 & ua_j \\ a_j^t u & 0 \end{pmatrix}, \quad u = \mathrm{diag}(+1, -1, \ldots, -1). \tag{68}$$

In [8] it is shown how these linear problems may be viewed as a direct and inverse scattering problem for the GWE and GSGE. Namely the direct and inverse problem may be solved for matrix potentials, depending on the orthogonal matrix a, tending to the identity sufficiently fast in certain "generic" directions. It should be noted that solving the n-dimensional GWE and GSGE reduces to the study of the scattering and inverse scattering associated with a coupled system of n one-dimensional o.d.e.'s. This is in marked contrast to other attempts described earlier to isolate solvable (local) multidimensional nonlinear evolution equation which are compatibility conditions of two Lax-type operators, e.g.,

$$L\psi = \lambda\psi \tag{69}$$

$$\psi_t = M\psi \tag{70}$$

where L is a partial differential operator with the variable t entering only parametrically. Although as we have seen nonlinear evolution equations in three independent variables can be associated with such Lax pairs (e.g. the K-P, Davey-Stewartson, three wave interaction equations, etc.) little progress via this route has been made in more than three dimensions. As discussed earlier one has to overcome a serious constraint inherent in the scattering/inverse scattering theory for higher dimensional partial differential operators in order to be able to isolate associated solvable nonlinear equations, i.e. the scattering data generally satisfies a nonlinear equation (eq. (42)). The analysis associated with the GWE and GSGE avoids these difficulties since the GWE and GSGE problems are simply a compatible set of nonlinear one-dimensional o.d.e.'s.

The results in ref. [8] demonstrate that the initial value problem is posed with given data along lines and not on (n-1) dimensional manifolds.

Similar ideas apply to certain *n*-dimensional extensions of the so-called anti-self-dual Yang-Mills equations (SDYM) [9]. In two complex variables the self-dual Yang Mills equations take the form (see [10])

$$\frac{\partial}{\partial \bar{x}_1}\left(\Omega^{-1}\frac{\partial \Omega}{\partial x_1}\right) + \frac{\partial}{\partial \bar{x}_2}\left(\Omega^{-1}\frac{\partial \Omega}{\partial x_2}\right) = 0, \tag{71}$$

where Ω is a positive matrix valued function of $(x_1,x_2)\in\mathbb{C}^2$. Alternatively SDYM takes the form

$$\frac{\partial A_1}{\partial \bar{x}_1} + \frac{\partial A_2}{\partial \bar{x}_2} = 0 \tag{72}$$

$$\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} + [A_1, A_2] = 0, \tag{73}$$

where

$$A_j = -\Omega^{-1}\frac{\partial \Omega}{\partial x_j} \tag{74}$$

The SDYM may be obtained via the compatibility condition of the following linear system

$$\begin{aligned} \frac{\partial m}{\partial x_1} - z\frac{\partial m}{\partial \bar{x}_2} &= A_1 m \\ \frac{\partial m}{\partial x_2} + z\frac{\partial m}{\partial \bar{x}_1} &= A_2 m \end{aligned} \tag{75}$$

multidimensional extensions may be obtained. For example, consider the linear system

$$D_z^j m(x,z) = A_j(x)m(x,z),\ j = 1,\cdots n \tag{76}$$

$$D_z^j = \frac{\partial}{\partial x_j} + zs_j\frac{\partial}{\partial \bar{x}_{j+1}} \tag{77}$$

and

$$x_{n+1} = x_1,\ s_j = (-1)^j.$$

Compatibility (commutativity) implies:

$$D_z^i A_j - D_z^j A_i + [A_i, A_j] = 0 \tag{78}$$

$$\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} + [A_i, A_j] = 0 \tag{79}$$

$$s_j \frac{\partial A_i}{\partial \bar{x}_{j+1}} - s_i \frac{\partial A_j}{\partial \bar{x}_{i+1}} = 0. \tag{80}$$

A potential Ω may be introduced as before: (81)

$$A_j = \Omega^{-1} \frac{\partial \Omega}{\partial x_j}$$

to obtain

$$s_j \frac{\partial}{\partial \bar{x}_{j+1}} \left(\Omega^{-1} \frac{\partial \Omega}{\partial x_i}\right) - s_i \frac{\partial}{\partial \bar{x}_{i+1}} \left(\Omega^{-1} \frac{\partial \Omega}{\partial x_j}\right) = 0 . \tag{82}$$

Clearly when n=2 this system reduces to the classical SDYM equation.

Solutions to these equations may be constructed via the $\bar{\partial}$ method. Define

$$D_z^j = L_1^j + zL_2^j \tag{83}$$

with

$$L_1^j = \frac{\partial}{\partial x_j}, \quad L_2^j = s_j \frac{\partial}{\partial \bar{x}_{j+1}} \quad .$$

We shall show that the $\bar{\partial}$ integral equation

$$m(x,z) = I + \frac{1}{2\pi i} \iint \frac{(mV)(x,\zeta)}{\zeta - z} \, d\zeta \cdot d\bar{\zeta} \tag{84}$$

satisfies (76). Operating on (84) with D_z^j yields,

$$D_z^j m = \frac{1}{2\pi i}\int \frac{(L_1^j m)V + m(L_1^j V)}{\zeta - z} d\zeta \cdot d\bar{\zeta} + J \tag{85}$$

where

$$J = \frac{1}{2\pi i}\int \frac{zL_2^j(mV)}{\zeta - z}\, d\zeta \cdot d\bar{\zeta}$$

$$= -\frac{1}{2\pi i}\int L_2^j(mV)d\zeta \cdot d\bar{\zeta}$$

$$+ \frac{1}{2\pi i}\int \frac{\zeta L_2^j(mV)}{\zeta - z}\, d\zeta \cdot d\bar{\zeta}. \tag{86}$$

Putting (85), (86) together gives

$$D_z^j m = \tilde{A}_j + \frac{1}{2\pi i}\int \frac{(D_\zeta^j m)V + m(D_\zeta^j V)}{\zeta - z}\, d\zeta \cdot d\bar{\zeta} \tag{87}$$

where

$$\tilde{A}_j(x) = -\frac{1}{2\pi i}\int L_2^j(mV)d\zeta \cdot d\zeta = -\frac{1}{2\pi i}\, s_j \frac{\partial}{\partial \bar{x}_{j+1}} \int (mV)d\zeta \cdot d\bar{\zeta}\ . \tag{88}$$

We shall require $V(x,z)$ to satisfy

$$D_z^j V = 0 \tag{89}$$

in which case using (84) in (87) by writing

$$\tilde{A}_j = \tilde{A}_j(m - \frac{1}{2\pi i}\int \frac{mV}{\zeta - z} d\zeta \cdot d\bar{\zeta}) \tag{90}$$

we find

$$(D_z^j m - \tilde{A}_j m) = \frac{1}{2\pi i}\int \frac{((D_\zeta^j m) - \tilde{A}_j(x)m)V}{\zeta - z} d\zeta \cdot d\bar{\zeta}. \qquad (91)$$

For V suitably chosen (84) has a unique solution in which case

$$D_z^j m - \tilde{A}_j m = 0. \qquad (92)$$

Thus $\tilde{A}_j = A_j$ and solutions of the extended SDYM are obtained.

The condition (89) is satisfied if we take $V(x,z) = V(u(x),z)$, with $u_j(x) = zx_j + s_{j+1}\bar{x}_{j+1}$ and V holomorphic in the u_j. Then

$$D_z^j V = \left(\frac{\partial}{\partial x_j} + zs_j \frac{\partial}{\partial \bar{x}_{j+1}}\right) V(u_i, \ldots u_n, z)$$

$$= \sum_{\ell=1}^{n} V'(u_\ell, z)(z\delta_{j\ell} + s_j s_{j+1} z\delta_{j\ell}) = 0 \qquad (93)$$

by virtue of $s_j = (-)^j$. In ref. [9] other examples of multidimensional extensions of SDYM and a rigorous derivation of the foregoing is given.

ACKNOWLEDGEMENTS

I am most pleased to acknowledge the many crucial contributions of my colleagues: A.S. Fokas, D. Bar Yaacov, A.I. Nachman, R. Beals and K. Tenenblat. This work was supported in part by the National Science Foundation under grant number DMS-8202117, the Office of Naval Research under grant number N00014-76-C-0867, and the Air Force Office of Scientific Research under grant number AFOSR-84-0005.

RERFERENCES

[1] Ablowitz, M.J. and Segur, H., "Solitons and the Inverse Scattering Transform", SIAM Appl. Math., Phila., PA, 4 (1981).

[2a] Ablowitz, M.J. and Fokas, A.S., "Comments on the Inverse Scattering Transform and Related Nonlinear Evolution Equations", Lect. Notes in Phys., 189, Proc. CIFMO School and Workshop held at Oaxtepec, Mexico (K.B. Wolf, ed.), Springer, 1982.

[2b] Beals, R. and Coifman, R., Commun. Pure Appl. Math., 1984, pp. 39-90.

[2c] Beals, R., The Inverse Problems for Ordinary Differential Operators on the Line; to appear, Amer. J. of Math.

[2d] Shabat, A.B., Func. Annal and Appl. 9, (1975); Diff. eq. XV (1979) 1824.

[2e] Caudrey, P., Physica 6D (1982) 51.

[3a] Manakov, S.V., Phys. 3D, 420 (1981).

[3b] Fokas, A.S. and Ablowitz, M.J. Stud. Appl. Math. 69, 211 (1983).

[4a] Ablowitz, M.J., BarYaacov, D., and Fokas, A.S., Stud. Appl. Math. 69, 135, (1983).

[4b] Fokas, A.S. and Ablowitz, M.J., The Inverse Scattering Transform for Multidimensional (2+1) Problems, Lect. Notes in Phys., 189, Proc. CIFMO School and Workshop held at Oaxtepec, Mexico, 1982, K.B. Wolf, ed.

[5a] Nachman, A.I. and Ablowitz, M.J., Stud. Appl. Math. 71, 243-250, (1984).

[5b] Ablowitz, M.J. and Nachman, A.I., Physica 18D, 223 (1986).

[5c] Nachman, A.I. and Lavine, R., On the Inverse Scattering Transform for the n-Dimensional Schrodinger Operator, to be published as Proceedings of conference on Nonlinear Evolution Equations, Solitons and the Inverse Scattering Transform, held at Mathematische Forschungsistitut, Oberwolfach W. Germany.

[6a] Nachman, A.I. and Ablowitz, M.J., Stud. Appl. Math. 71, 251-262, (1984).

[6b] Fokas, A.S., Phys. Rev. Lett., 57, No. 2, 159 (1986).

[6c] Fokas, A.S., J. Math. Phys. 27 (7), 1737-1746 (1986).

[7a] Beals, R., and Coifman, R., Multidimensional Inverse Scattering on Nonlinear PDE, Proc. Symposium Pure Math, 43, 45 (1985).

[7b] Beals, R. and Coifman, R., Physica 18D, 242, (1986).

[8] Ablowitz, M.J., Beals, R., and Tenenblat, K., Stud. Appl. Math., 74, 177-203 (1986).

[9] Ablowitz, M.J., Costa, D.G., and Tenenblat, K., Solutions of Multidimensional Extensions of the Anti-Self-Dual Yang-Mills Equations, preprint, INS #66, 1986.

[10] Pohlmeyer, K., Commun. Math. Phys. 72, 37-47, (1980).

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